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1993 J. Phys. A: Math. Gen. 26 591

(http://iopscience.iop.org/0305-4470/26/3/020)

J. Phys. A: Math. Gen. 26 (1993) 591–607. Printed in the UK

Poincaré coherent states: the two-dimensional massless case

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Received 5 January 1992, in final form 5 October 1992

Abstract. Coherent states for the positive mass representations of the Poincaré group in $1+1$ dimensions have been obtained previously, using the fact that these representations are square integrable moduli of the subgroup of time translations. Here the method is extended by combining sections from the coset space into the group with homeomorphisms of the coset space (these maps are called quasi-sections). Then the generalized construction is applied to the zero mass representations of the $(1+1)$-dimensional Poincaré group, which are square integrable moduli of a subgroup of light-like translations. The resulting coherent states, indexed as before by points in phase space, yield a resolution of the identity in the Krein space of the zero mass representations (the first explicit example of such a structure), and it turns out that they coincide with the familiar wavelets based on the '$ax+b$' group.

1. Introduction

In view of their importance in all branches of physics, coherent states have gradually been defined in situations of increasing generality. In terms of group theory, this means extending the construction of coherent states from the Weyl–Heisenberg group (the so-called canonical coherent states) to larger and larger groups—the basic ingredient being a square integrable unitary representation (see [1] and [2] for a review).

However, groups of the form of a semidirect product $G = V \wedge L$, with V a vector space and $L \subset Gl(V)$, are not amenable to this treatment: the relevant, Wigner-type, representations are not square integrable. Typical in this respect are the Euclidean, Galilei and Poincaré groups; precisely the relativity groups most commonly used in physics! This situation prompted Ali, Gazeau and one of us [3–6] to extend the coherent states machinery to representations which are only square integrable on a homogeneous space G/H, where H is a closed subgroup of G (and even further, to a setup without any reference to group representations at all). The central point of the construction they proposed is the choice of a section $\sigma : G/H \to G$ such that one can define a generalized square-integrability condition (see equation (2.1)).

Using this notion, one may build up sets of coherent states for the various *representations of the relativity groups; in particular coherent states for massive* representations ($m > 0$) of the Poincaré group in $1+1$ space–time dimensions are exhibited in [3–6] (everything works as well in the 'physical' $(1+3)$-dimensional case).

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In this paper we want to construct coherent states for *massless* representations ($m = 0$) of the two-dimensional the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$. This is the first step toward a similar analysis for the (infinite dimensional) conformal group. However, in two dimensions there is a sharp difference between the massless and the massive case. The massive representations, which live on the corresponding mass shell, are irreducible and unitary. In the same way, there exist two massless representations, both irreducible and unitary, living respectively on the two pieces of the light-cone, $k^0 = \pm|k^1|$, and corresponding to the solutions of the *classical* wave equation $\Box W = 0$ [7] (note that the representation spaces do not contain the Schwartz space as a dense subspace). However these representations are inadequate for a relativistic *quantum* theory. Indeed, as is well known [8–12], the massless two-point function does not satisfy the positivity condition of a Wightman quantum field theory (QFT), because of the infrared problem which is more severe in two dimensions than in four. Hence, for obtaining a fully satisfactory massless QFT in two dimensions, one must enlarge the representation space and use an indefinite metric space. The resulting massless representation of $\mathcal{P}_+^\uparrow(1,1)$ is non-decomposable and non-unitary, and the previous unitary representations may be derived from it by taking an appropriate quotient (we recall in the appendix the explicit construction of the relevant representations, massive and massless, according to [12–15]). The most interesting features of the corresponding quantum field theory are precisely linked to the infrared properties of this indefinite metric representation, and are completely lost when one restricts it to the unitary ones. In particular, these infrared properties are at the basis of the fermion bosonization, the possibility of exact solution of the Schwinger and the Thirring models, and almost all phenomena regarding the exactly soluble models of two-dimensional quantum field theory [10].

The same infrared divergences also cause a failure of the construction of coherent states given in [3–6], when applied to the present case: for all the 'natural' sections, similar to those used in the massive case [6], there is *no* vector that satisfies the integrability condition (2.1). In other words, the formalism of [3–6] is too restrictive, and, to cover the massless case, we are forced to extend it.

The general construction that we will use is presented in section 2. As compared with [6], the new aspects are twofold. First, following [16], we do not assume that the coset space $X = G/H$ has a G-invariant measure, but only a quasi-invariant one (this permits us to treat the case of a non-unimodular subgroup H, and some infinite-dimensional groups as well [16]). Second, we allow sections of the principal bundle $(G, \pi, G/H, H)$ to be combined with homeomorphisms of the base manifold $X = G/H$, (2.8). We will call the resulting maps 'quasi-sections'. This essentially means that we are now considering sections in a certain induced (or pull-back) bundle [17] defined by the given homeomorphism of the base manifold. Then, as before, when the appropriate integrability condition is satisfied, the construction yields an overcomplete set of vectors, called *quasi-coherent states.* They have all the nice properties usually associated with coherent states and needed for applications. However, for reasons of consistency, we reserve the name *coherent states* to those that are obtained by transporting a fixed vector (or set of vectors) over X under the action of G, in a *covariant* way. This definition covers all cases treated previously in the literature [1, 2]. Clearly, if the quasi-section is not a genuine section, the resulting states will be only quasi-coherent, but for applications this makes little difference.

Thus, the construction of (quasi-)coherent states amounts to finding a suitable homogeneous space $X = G/H$ for which an admissible quasi-section exists, i.e. a quasi-section which verifies the square integrability condition for the representation considered (see (2.8)). Since the coherent states will be indexed by points of X, a natural question is whether the homegeneous space X has a physical meaning (at least in the case of the relativity groups). Quantization arguments [16, 18] suggest that X should be a *phase space* for the system at hand. This choice has several advantages. Contrary to their space–time relatives, phase space realizations of quantum mechanics are very well adapted to the description of localization properties and of the measurement process [19]. In addition it is a nice way of recovering the classical character of coherent states, since phase space is the natural arena of classical mechanics. Now there is a distinguished class of phase spaces associated to a given Lie group G; they are the orbits of G in $\mathfrak{g}^*$ (the dual of the Lie algebra $\mathfrak{g}$) under the coadjoint action [20]. Such orbits have a natural symplectic (even Kähler) structure, a unique measure invariant under the action of G (up to normalization) and they correspond to unitary representations of G by the familiar Mackey–Kirillov construction. Going back to our coherent states problem, each coadjoint orbit of G may be identified with a corresponding homogeneous space G/H, H being the stabilizer of a given point of the orbit. In section 3 we briefly analyse the coadjoint orbits of $\mathcal{P}_+^\uparrow(1{,}1)$. Each non-degenerate orbit corresponds to a unique unitary irreducible representation, massive or massless, as described in section 6.

The new formalism we are presenting here is much more flexible than the old one, and it also has interesting consequences in the massive case; new systems of coherent states are constructed and discussed in section 4. On the other hand it permits the construction of massless Poincaré coherent states, as we show explicitly in section 5. As a byproduct, we obtain two interesting results. First, as expected, the massless coherent states generate a resolution of the identity, but in an *indefinite metric* space, namely the Krein representation space. As far as we know, this is the first explicit example of such a resolution, a result of intrinsic mathematical interest. To be sure, the case met here is the simplest one, in which the Krein space is a Pontriagin space. Nevertheless our result already suggests the possibility of extending the construction of coherent states to non-unitary representations of groups. It also opens a new direction of investigation related to gauge theories, since, as is well known, the latter may be quantized in a covariant way only with an indefinite metric. The massless scalar field in two dimensions treated here is indeed the simplest example of a gauge theory with local gauge invariance. Second, if one chooses a suitable quasi-section of the principal bundle $(\mathcal{P}_+^\uparrow(1,1), \pi, \mathcal{P}_+^\uparrow(1,1)/L_{l(r)}, L_{l(r)})$ ($L_{l(r)}$ is the subgroup of left (right) light-like translations), the massless coherent states of $\mathcal{P}_+^\uparrow(1{,}1)$ coincide with *wavelets*. The latter are usually defined as coherent states associated with the affine group '$ax + b$' of the line. We see here that they are also *massless* coherent states for the Poincaré group. This fact opens a new and major range of applications for wavelets; two-dimensional quantum field theory. Actually this connection between Poincaré coherent states and wavelets is not so surprising, since the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$ and the affine group '$ax + b$' have the same half-plane as phase space. Indeed the link has been noticed in the literature [7, 21], although not always explicitly.

2. Coherent states: the general construction

In a previous series of papers, Ali, Gazeau and one of us [3–6] introduced a generalized notion of square integrability for a group representation, which is the key point for constructing coherent states when the usual methods fail [1, 2]. However, for the massless representations of the Poincaré group, we have to extend that method one step further. Let us resume the main features of the construction given by [3–6].

Let G be a locally compact group and H a closed subgroup of G. Consider the coset space $X = G/H$ and let ν be a (quasi)-invariant measure on X [22]; such a measure always exists and is unique, up to equivalence (only the case of an invariant measure was considered in [3–6]; the general case is discussed in [16]). Denote by $\mathrm{d}\nu_g(x) = \mathrm{d}\nu(g^{-1}x)$ the translated measure and by $\lambda(g,x) = \mathrm{d}\nu_g(x)/\mathrm{d}\nu(x)$ the corresponding Radon–Nikodym derivative.

Let $U(g)$ be a unitary irreducible representation of G on a Hilbert space $\mathcal{H}$ and let σ: $G/H \to G$ be a section of the canonical principal bundle $(G, \pi, G/H, H)$. The representation U is called *square integrable mod* (H, σ) if for some $\zeta \in \mathcal{H}$ the following integral converges and is strictly positive for all $\phi \in \mathcal{D} \subset \mathcal{H}$, $\mathcal{D}$ dense

$$I_\sigma(\zeta,\phi) = \int_X |\langle U(\sigma(x))\zeta, \phi\rangle_{\mathcal{H}}|^2 \lambda(\sigma(x),x)\,\mathrm{d}\nu(x) < \infty \tag{2.1}$$

where $\lambda(\sigma(x),x)\,\mathrm{d}\nu(x) = \mathrm{d}\nu_{\sigma(x)}(x)$; in other words, if the quadratic form (2.1) defines a positive *invertible* operator A_σ^ζ

$$0 < I_\sigma(\zeta,\phi) = \langle \phi, A_\sigma^\zeta \phi\rangle < \infty \qquad \forall \phi \in \mathcal{D}. \tag{2.2}$$

When this condition is satisfied, the family of vectors:

$$\mathfrak{S}_\sigma^\zeta = \{\zeta_{\sigma(x)} = \sqrt{\lambda(\sigma(x),x)}\,U(\sigma(x))\zeta,\ x \in X\} \tag{2.3}$$

is a family of *coherent states* with all the expected properties. All those vectors ζ for which the integral (2.1) converges are called *admissible*, and the section σ itself is said to be *admissible* for U. If ζ itself is in $\mathcal{D}$, we define

$$c_\sigma(\zeta) = I_\sigma(\zeta,\zeta). \tag{2.4}$$

Three remarks are in order here:

(i) the inclusion of the factor $\lambda(\sigma(x),x)$ in (2.1) guarantees the covariance of the admissibility condition: if the section σ is admissible for U, so is every section σ_g obtained from σ by the natural action of G on X [16]. If the measure ν is invariant, $\lambda(g,x) \equiv 1$ and one gets back the situation of [3–6].

(ii) The factor $\lambda(\sigma(x),x)$ also implies that the admissibility condition (2.1) depends only on the equivalence class of the measure ν, indeed if ν' is another quasi-invariant measure, it is necessarily equivalent to ν, i.e. $\mathrm{d}\nu'(x) = \alpha(x)\,\mathrm{d}\nu(x)$. then one has

$$\lambda'(g,x) = \frac{\mathrm{d}\nu'_g(x)}{\mathrm{d}\nu'(x)} = \frac{\alpha(g^{-1}x)}{\alpha(x)}\lambda(g,x) \tag{2.5}$$

and therefore

$$\lambda'(\sigma(x),x)\,\mathrm{d}\nu'(x) = \alpha(\sigma(x)^{-1}x)\lambda(\sigma(x),x)\,\mathrm{d}\nu(x). \tag{2.6}$$

Since $x = \sigma(x)x_0$ for some base point $x_0 \in X$, it follows that $\alpha(\sigma(x)^{-1}x) = \alpha(x_0)$ is a constant and the integrability condition (2.1) is independent on the choice of the quasi-invariant measure.

(iii) The definition (2.1) is, in fact, slightly more general than the one given in [3–6] since it allows the possibility of 'resolving' an unbounded operator for $\mathcal{D} \neq \mathcal{H}$ (see later; this possibility was also mentioned in [6, proposition 2.2]).

We now spell out the square-integrability condition which generalizes the condition (2.1) and that we will need later in specific cases. As before, let G be a locally compact topological group, H a closed subgroup of G, $U(g)$ a unitary irreducible representation of G in a Hilbert space $\mathcal{H}$. We call a *quasi-section* of the principal bundle $(G, \pi, G/H, H)$ a map σ_f which satisfies the following condition:

$$\sigma_f : G/H \to G \qquad \text{and} \qquad \pi \cdot \sigma_f = f \tag{2.7}$$

where f is a homeomorphism of G/H into itself. It is clear that the quasi-section σ_f is the product $\sigma \circ f$ of a genuine section and a homeomorphism of the base manifold.

We say that the representation $U(g)$ is square integrable mod(H, σ_f) if for some $\zeta \in \mathcal{H}$ the following integral converges and is strictly positive for all $\phi \in \mathcal{D} \subset \mathcal{H}$, $\mathcal{D}$ dense:

$$I_{\sigma_f}(\zeta, \phi) = \int_X |\langle U(\sigma_f(x))\zeta, \phi\rangle_{\mathcal{H}}|^2 \, \mathrm{d}\nu(x). \tag{2.8}$$

Looking at σ_f as $\sigma \cdot f$, where $\sigma = \sigma_f \cdot f^{-1} : G/H \to G$ is a genuine section of the principal bundle $G \to G/H$; performing the change of variables $y = f(x)$ in (2.8), one obtains an integral similar to the previous one (2.1), but with $\mathrm{d}\nu(x)$ replaced by $\mathrm{d}\mu(x) = (f^{-1})'(x)\, \mathrm{d}\nu(x)$. The resulting measure μ is not usually the $\sigma(x)$-translate of ν, since $(f^{-1})'(x) \neq \lambda(\sigma(x), x)$ in general. Therefore we shall call the vectors $\zeta_{\sigma(x)} = U(\sigma \cdot f(x))\zeta$ a family of *quasi-coherent states*. They enjoy all the nice properties of the coherent states (2.3) except covariance, in the sense that the two properties (i) and (ii) do not necessarily hold any more.

In fact, if we replace in (2.1) $\mathrm{d}\nu_{\sigma(x)}(x) = \lambda(\sigma(x), x)\, \mathrm{d}\nu(x)$ by an *arbitrary* quasi-invariant measure $\mathrm{d}\mu(x) = \alpha(x)\, \mathrm{d}\nu(x)$, with $\alpha(x) \neq \lambda(\sigma(x), x)$, we still do have a useful overcomplete set of vectors, namely

$$\mathfrak{S}'^{\zeta}_{\sigma} = \{\zeta'_{\sigma(x)} = U(\sigma(x))\zeta,\ x \in X\}. \tag{2.9}$$

As before those vectors will be called quasi-coherent states [3–6]. However, it may happen that the integral with the covariant factor $\lambda(\sigma(x), x)$ instead of $\alpha(x)$, actually diverges; this means that there are no 'true' coherent states associated with the section σ, and the given representation.

To conclude with generalities, it is worthwhile to mention that one still faces the fact that the operator $A^{\zeta}_{\sigma_f}$ which is 'resolved' by the family of coherent states thus constructed is not necessarily a multiple of the identity. However it is always possible to recover a genuine resolution of the identity by introducing a weighting operator [4–6, 23, 24]. Thus in all cases one ends up with quasi-coherent states.

3. Coadjoint orbits of $\mathcal{P}^{\uparrow}_{+}(1,1)$

In this section we briefly examine the coadjoint orbit structure of $\mathcal{P}^{\uparrow}_{+}(1,1)$. This is important because, as we will see, the coherent states that we are going to construct will be labelled by points of suitable coadjoint orbits (phase spaces), and, as was briefly mentioned in the introduction, this fact makes them central objects for quantization procedures. The method that we will apply follows that of [25] and may be applied in all cases in which the Lie group is a semidirect product (see also [26]).

Let L be a Lie group and V a real vector space. Given a representation of L on V, we can construct the semidirect product $G = V \wedge L$. Elements g of G are written as follows

$$g = (v, \Lambda) = \begin{bmatrix} \Lambda & v \\ 0 & 1 \end{bmatrix} \tag{3.1}$$

where $v \in V, \Lambda \in L \subset Gl(V)$. Elements $\mathcal{G}$ of the Lie algebra of G, denoted $\mathfrak{g} = V \oplus \mathfrak{l}$, are then written as

$$\mathcal{G} = (w, Y) = \begin{bmatrix} Y & w \\ 0 & 0 \end{bmatrix} \tag{3.2}$$

where $w \in V, Y \in \mathfrak{l} \subset gl(V)$. The adjoint representation of G on $\mathfrak{g}$ is defined by

$$\mathrm{Ad}(g)\mathcal{G} = g\mathcal{G}g^{-1} = \begin{bmatrix} \Lambda Y \Lambda^{-1} & \Lambda w - \Lambda Y \Lambda^{-1} v \\ 0 & 0 \end{bmatrix}. \tag{3.3}$$

Let us write a generic element of $\mathfrak{g}^* = V^* \oplus \mathfrak{l}^*$ as (ξ, λ) with $\xi \in V^*, \lambda \in \mathfrak{l}^*$. Then, denoting the coadjoint representation of G as $\mathrm{Ad}^\sharp$, we obtain

$$\begin{aligned} \langle \mathrm{Ad}^\sharp(g)(\xi, \lambda), \mathcal{G} \rangle_{\mathfrak{g}^*, \mathfrak{g}} &= \langle (\xi, \lambda), \mathrm{Ad}(g)^{-1}\mathcal{G} \rangle_{\mathfrak{g}^*, \mathfrak{g}} \\ &= \langle \xi, \Lambda^{-1} Y v + \Lambda^{-1} w \rangle_{V^*, V} + \langle \lambda, \Lambda^{-1} Y \Lambda \rangle_{\mathfrak{l}^*, \mathfrak{l}}. \end{aligned} \tag{3.4}$$

For $\xi \in V^*$ and $v \in V$ define $\xi \odot v \in \mathfrak{l}^*$ by

$$\langle \xi \odot v, Y \rangle_{\mathfrak{l}^*, \mathfrak{l}} = \langle \xi, Y v \rangle_{V^*, V}. \tag{3.5}$$

Then we finally get the formula

$$\mathrm{Ad}^\sharp(g)(\xi, \lambda) = ((\Lambda^{-1})^* \xi, \mathrm{Ad}^\sharp{}_{\mathfrak{l}}(\Lambda)\lambda + (\Lambda^{-1})^* \xi \odot v). \tag{3.6}$$

Here $\mathrm{Ad}^\sharp_{\mathfrak{l}}$ indicates the coadjoint representation of L on $\mathfrak{l}^*$, and $(\Lambda^{-1})^*$ the representation of L on V^* contragredient to the original one on V. For more details on this subject see [20, 25, 26].

Let us apply (3.6) in the concrete case $G = \mathcal{P}_+^\uparrow(1,1) = M^2 \wedge \mathrm{SO}_0(1,1)$ (see also [27] and, for the corresponding analysis in the $(1+3)$-dimensional case, [26, 28, 29]). We may identify M^2 with its own dual by using the Minkowski inner product. Elements of $\mathrm{SO}_0(1,1) = \mathcal{L}_+^\uparrow$ will be parametrized in the following way

$$\Lambda_p = \begin{bmatrix} p^0 & p \\ p & p^0 \end{bmatrix} \qquad \text{where } p^0 = \sqrt{p^2 + 1}. \tag{3.7}$$

The corresponding Lie algebra $so(1,1) \cong \mathbb{R}$ is generated by the element

$$Y_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.8}$$

This fact, together with (3.7), implies that $\mathrm{Ad}_{|}^{\#}(\Lambda_p)\lambda = \lambda$ and therefore the coadjoint orbits of $\mathrm{SO}_0(1,1)$ are given by the following formula:

$$\mathrm{Ad}^{\#}(g)(\xi,\lambda) = (\Lambda_p\xi, \lambda + \langle\Lambda_p\xi, Y_0 v\rangle_{M^2}) \tag{3.9}$$

where g is as in (3.1) , with $\Lambda_p \in \mathcal{L}_+^\uparrow, \xi, v \in M^2$ and $\langle\cdot,\cdot\rangle_{M^2}$ denotes the Minkowski inner product. Now we can easily identify the coadjoint orbits of $\mathcal{P}_+^\uparrow$(1,1): they are given by the following families of hyperbolic cylinders ($m > 0$):

$$\xi^0 = \pm\sqrt{\xi^{1^2} + m^2} \qquad \xi^1 = \sqrt{\xi^{0^2} + m^2} \tag{3.10}$$

the four half-planes

$$\xi^0 = \pm\xi^1 \qquad \xi^0 > 0 \qquad \xi^0 = \pm\xi^1 \qquad \xi^0 < 0 \tag{3.11}$$

and the degenerate orbits consisting of a single point

$$\lambda = \text{constant} \qquad \xi = 0. \tag{3.12}$$

The non-degenerate coadjoint orbits may be interpreted as classical phase spaces corresponding to elementary systems having $\mathcal{P}_+^\uparrow$(1,1) as relativity group [20].

As we will see, they are particularly suited for the construction of systems of coherent states and it turns out that in each such system the states are indexed by the points of a certain coadjoint orbit. Every orbit in turn may be identified with a homogeneous space $X = \mathcal{P}_+^\uparrow(1,1)/H$, where H is the stabilizer of a given point of the orbit under the coadjoint action.

4. Application: massive coherent states for $\mathcal{P}_+^\uparrow$(1,1)

For a better understanding of the role of the square-integrability condition that we have introduced in (2.8), we now return to the specific case of the Poincaré group (the representations of $\mathcal{P}_+^\uparrow$(1 ,1) corresponding to a particle of mass m are displayed in (A.13)). We have to select a homogeneous space to apply the previous construction. It is natural to choose, as the homogeneous space, the corresponding classical phase space [3,30], i.e. the coadjoint orbit $\Gamma_{m>0} : \xi^0 = \sqrt{\xi^{1^2} + m^2}$. Let us examine more closely the case $m = 1$. The corresponding orbit Γ_1 is generated by the point $\lambda = 0, \xi = (1,0)$. Since the stabilizer of this point is the subgroup T of time translations we obtain that Γ_1 is isomorphic to the homogeneous space $\mathcal{P}_+^\uparrow/T$ [20]. Γ_1 is the hyperbolic cylinder of equation $\xi^0 = \sqrt{\xi^{1^2} + 1}$. We may choose different parametrizations of this orbit; the simplest one is that obtained by projection on the $\xi^0 = 0$ plane, which is given by

$$(\tau, p) \qquad \text{with } \tau = \lambda,\ p = \xi^1. \tag{4.1}$$

These coordinates do *not* have the meaning of classical position and momentum, contrary to the coordinates (q,p) used in [3–6]. The relation between the two sets reads as follows

$$q = \tau/p^0 \qquad p = p \qquad \text{with } p^0 = \sqrt{p^2+1}. \tag{4.2}$$

Nevertheless they are useful in the construction that follows. The coadjoint action may now be rewritten in the following way

$$(\tau, p) \to (\tau', p') = (a, \Lambda_k)(\tau, p) \tag{4.3}$$

with

$$\tau' = \tau + \langle \Lambda_k p, Y_0 a \rangle_{M^2} \qquad p' = pk^0 + kp^0. \tag{4.4}$$

Thus the coadjoint action on Γ_1 may be identified with the left action [4] of $\mathcal{P}_+^\uparrow(1,1)$ on $\mathcal{P}_+^\uparrow(1,1)/T$.

The orbit Γ_1 carries a unique (up to normalization) measure invariant with respect to the transformations (4.4). In the parametrization (4.1), this invariant (Liouville) measure reads as

$$\mathrm{d}\nu(\tau, p) = \frac{\mathrm{d}\tau \mathrm{d}p}{p^0} (= \mathrm{d}q\, \mathrm{d}p). \tag{4.5}$$

At this point we have to choose admissible quasi-sections $\sigma_f = \sigma \cdot f : \Gamma_1 \to \mathcal{P}_+^\uparrow(1,1)$, for suitable homeomorphisms $f : \Gamma_1 \to \Gamma_1$; each choice will lead to a set of quasi-coherent states. We give three examples of such quasi-sections.

4.1. Natural (or naive) quasi-section

This quasi-section is defined by

$$\sigma_n(\tau, p) = ((0, \tau), \Lambda_p). \tag{4.6}$$

It turns out that every vector of $\mathcal{H}_{m=1} \equiv L^2(\mathcal{V}_{m=1}^+, \mathrm{d}k/k^0)$ (for simplicity we write $k \equiv k^1$) is admissible and the following equality holds (in the sense of quadratic forms):

$$\int_{\Gamma_1} |U_{m=1}(\sigma_n(\tau,p))\zeta\rangle\langle U_{m=1}(\sigma_n(\tau,p))\zeta|\, \mathrm{d}\nu(\tau,p) = C_\zeta H^{-1} \tag{4.7}$$

where H is the energy operator, defined on a dense subset of $\mathcal{H}_{m=1}$ by

$$(H\psi)(k) = k^0 \psi(k) \tag{4.8}$$

H^{-1} is its (bounded) inverse and C_ζ is a constant depending on ζ. Since H is unbounded, this quasi-section does not yield a frame [5]. Notice that, if

$$\zeta \in \mathcal{D}(H^{1/2}) \tag{4.9}$$

then also $U_{m=1}(\sigma_n(\tau,p))\zeta \in \mathcal{D}(H^{1/2})$ and we obtain that the states

$$\zeta_{\tau,p}(k) \equiv (\sqrt{C_\zeta} H^{1/2} U_{m=1}(\sigma_n(\tau,p))\zeta)(k) = \sqrt{C_\zeta k^0}\, \mathrm{e}^{\mathrm{i}\tau k}\, \zeta(p^0 k - pk^0) \tag{4.10}$$

yield a *resolution of the identity.*

4.2. General Galilean quasi-section

The quasi-section σ_n is a particular case of the following general class of quasi-sections obtained from σ_0:

$$\sigma_{\mathrm{Gal}}(\tau,p) = ((0,\psi(p)\tau),\Lambda_{\rho(p)}) \tag{4.11}$$

where $\phi : (\tau,p) \mapsto (\psi(p)\tau,\rho(p))$ is a homeomorphism of Γ_1 onto itself. Such quasi-sections may be called 'Galilean' because they assign the value $a^0 = 0$ to every point (τ,p). For these general quasi-sections one may compute explicitly the class of admissible vectors and the corresponding 'resolved' operator.

4.3. Canonical section

This section is given by

$$\sigma_0(\tau,p) = ((0,\tau/p^0),\Lambda_p). \tag{4.12}$$

We call it 'canonical' because it is a genuine section of the principal bundle, and also because the coordinates $(\tau/p^0,p) = (q,p)$ may have the interpretation of position and momentum. This section has been studied in [3, 6], where it was called β_0. In this case one can show that admissible vectors must satisfy again condition (4.9); under such condition we obtain the following weak identity

$$\int_{\Gamma_1} |U_{m=1}(\sigma_0(\tau,p))\zeta\rangle\langle U_{m=1}(\sigma_0(\tau,p))\zeta|\,\mathrm{d}\nu(\tau,p) = A_\zeta \tag{4.13}$$

where both A_ζ and A_ζ^{-1} are bounded operators on $\mathcal{H}_{m=1}$. Thus in this case we obtain a frame (in general not tight) [5].

5. Massless coherent states

Let us now pass to the construction of masslesss coherent states. Again we have to choose a suitable homogeneous space and a unitary irreducible representation of $\mathcal{P}_+^\uparrow(1,1)$ which correspond to the massless case. First we identify the classical phase space corresponding to a massless relativistic particle. A look at the structure of the coadjoint orbits shows that the ones corresponding to a massless particle are the half-planes

$$\Gamma_1 = \{\lambda \in \mathbb{R}, \xi \in M^2 : \xi^0 = -\xi^1, \xi^1 < 0\}: \qquad \text{left orbit} \tag{5.1}$$

$$\Gamma_r = \{\lambda \in \mathbb{R}, \xi \in M^2 : \xi^0 = \xi^1, \xi^1 > 0\}: \qquad \text{right orbit.} \tag{5.2}$$

Let us concentrate our attention on Γ_r. This orbit is generated by the point $\lambda = 0,\ \xi = (1,1)$. The stabilizer of this point is the subgroup of right light-like translations

$$L_r = \{g \in \mathcal{P}_+^\uparrow(1,1) : g = (a,1), a^0 = a^1\} \tag{5.3}$$

and therefore $\Gamma_r \simeq \mathcal{P}_+^\uparrow(1,1)/L_r$. A global parametrization of this orbit is again given by

$$(\tau, p) \qquad \text{with} \qquad \tau = \lambda \qquad p = \xi^1. \tag{5.4}$$

In terms of these coordinates the coadjoint action reads

$$\begin{aligned} &(\tau, p) \to (\tau', p') = (a, \Lambda_k)(\tau, p) \\ &\tau' = \tau + \langle \Lambda_k p, Y_0 a\rangle_{M^2} \qquad p' = (k^0 + k)p \end{aligned} \tag{5.5}$$

and the invariant measure becomes

$$\mathrm{d}\mu(\tau, p) = \mathrm{d}\tau\, \mathrm{d}p/p. \tag{5.6}$$

As for the choice of the representation, we consider the representation U_r on the Hilbert space $\mathcal{H}_r$ that is displayed in (A.26) (we write again $k \equiv k^1$). Now we must choose a quasi-section $\sigma : \Gamma_r \to \mathcal{P}_+^\uparrow$ and try to construct coherent states out of it. First of all notice that the natural quasi-section $\sigma_n(\tau, p) = ((0, \tau), \Lambda_p)$ cannot do the work. Indeed it may be directly verified that there is no $\zeta \in \mathcal{H}_r$ such that

$$I_{\sigma_n}(\zeta, \phi) = \int_{\Gamma_r} |\langle U_r(\sigma_n(\tau, p))\zeta, \phi\rangle_{\mathcal{H}_r}|^2\, \mathrm{d}\mu(\tau, p) < \infty \tag{5.7}$$

(the integral is infrared divergent!). The reason for this fact is that one has

$$(U_r(\sigma_n(\tau, p))\zeta)(k) = \mathrm{e}^{\mathrm{i}\tau k}\zeta((p^0 - p)k). \tag{5.8}$$

Since $p > 0$ it follows that

$$0 < (p^0 - p) < 1 \tag{5.9}$$

and therefore the argument of the function ζ cannot be arbitrarily dilated. A well chosen quasi-section should have the following form

$$\sigma_\rho(\tau, p) = ((0, \tau), \Lambda_{\rho(p)}) \tag{5.10}$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}$ is an auxiliary bijective map. An interesting explicit form for the function ρ is the following one:

$$\rho(p) = \frac{1}{2p} - \frac{p}{2}. \tag{5.11}$$

The nice features of this function are due to the following fact

$$\sqrt{\rho^2(p) + 1} = \frac{1}{2p} + \frac{p}{2}. \tag{5.12}$$

Consequently we obtain

$$(U_r(\sigma_\rho(\tau, p))\zeta)(k) = \mathrm{e}^{\mathrm{i}\tau k}\zeta(pk) \qquad \zeta \in \mathcal{H}_r. \tag{5.13}$$

and therefore

$$I_{\sigma_\rho}(\zeta,\phi)=\int_{\Gamma_r}|\langle U_r(\sigma_\rho(\tau,p))\zeta,\phi\rangle_{\mathcal{H}_r}|^2\,\mathrm{d}\mu(\tau,p)=\int_0^\infty\frac{\mathrm{d}p}{p}\int_0^\infty\frac{\mathrm{d}k}{k^2}|\zeta(pk)|^2|\phi(k)|^2. \tag{5.14}$$

Now we apply the Fubini–Tonelli theorem and exploit the following change of variables:

$$u=kp\qquad\frac{\mathrm{d}u}{u}=\frac{\mathrm{d}p}{p}. \tag{5.15}$$

It follows that

$$I_{\sigma_\rho}(\zeta,\phi)=\int_0^\infty\frac{\mathrm{d}u}{u}|\zeta(u)|^2\int_0^\infty\frac{\mathrm{d}k}{k^2}|\phi(k)|^2. \tag{5.16}$$

Define the following operator on $\mathcal{H}_r$:

$$\mathcal{D}(H_r)=\left\{\phi\in\mathcal{H}_r;\int_0^\infty\,\mathrm{d}k\,k|\phi(k)|^2<\infty\right\} \tag{5.17}$$

$$(H_r\phi)(k)=k\phi(k)\qquad\phi\in\mathcal{D}(H_r). \tag{5.18}$$

H_r is an unbounded self-adjoint operator on $\mathcal{D}(H_r)\subset\mathcal{H}_r$ and $I_{\sigma_\rho}(\zeta,\phi)$ exists if and only if $\phi\in\mathcal{D}(H_r^{-1/2})$. In this case it follows that

$$\int_{\Gamma_r}|U_r(\sigma_\rho(\tau,p))\zeta\rangle\langle U_r(\sigma_\rho(\tau,p))\zeta|\,\mathrm{d}\mu(\tau,p)=C_\zeta H_r^{-1} \tag{5.19}$$

in the sense of quadratic forms. Thus, the vectors given in (5.13) constitute a set of massless coherent states for every $\zeta\in\mathcal{H}_r$, but the operator that is 'resolved' in (5.19) is not bounded and, moreover, its inverse is also unbounded. Because of these facts, this set of coherent states is more general than those considered in [6]. Let us exploit our freedom in the choice of quasi-sections (and bundle structure) to get a more appealing set of massless coherent states. The quasi-section that will do the work is given by

$$\sigma_r(\tau,p)=((0,\tau/p),\Lambda_{-\rho(p)}). \tag{5.20}$$

With the help of this quasi-section we get the following set of states:

$$(U_r(\sigma_r(\tau,p))\zeta)(k)=\mathrm{e}^{\mathrm{i}(\tau/p)k}\zeta(k/p)\qquad\zeta\in\mathcal{H}_r. \tag{5.21}$$

In this case the integral (2.1) becomes

$$I_{\sigma_r}(\zeta,\phi)=\int_0^\infty\,\mathrm{d}p\int_0^\infty\frac{\mathrm{d}k}{k^2}|\zeta(k/p)|^2|\phi(k)|^2. \tag{5.22}$$

By using the Fubini–Tonelli theorem again and the following change of variables

$$u=\frac{k}{p}\qquad\mathrm{d}u=-\frac{u^2}{k}\,\mathrm{d}p \tag{5.23}$$

we obtain

$$I_{\sigma_r}(\zeta,\phi) = \int_0^\infty \frac{du}{u^2}|\zeta(u)|^2 \int_0^\infty \frac{dk}{k}|\phi(k)|^2. \tag{5.24}$$

Therefore, vectors $\zeta \in \mathcal{H}_r$ are admissible for the quasi-section (5.20) if and only if they satisfy the condition $\zeta \in \mathcal{D}(H_r^{-1/2})$. If this condition is satisfied we obtain

$$\frac{1}{c_{\sigma_r}(\zeta)} \int_{\Gamma_r} |U_r(\sigma_r(\tau,p))\zeta\rangle\langle U_r(\sigma_r(\tau,p))\zeta| \, d\mu(\tau,p) = I \tag{5.25}$$

i.e. we get a genuine resolution of the identity! We call the states defined in equation (5.21) *right coherent states*. In a perfectly identical way we may construct a corresponding set of *left coherent states*. Points in Γ_l are parametrized by (τ, p), with $\tau \in \mathbb{R}$ and $p \in \mathbb{R}^-$; the quasi-section has exactly the same form as in (5.20):

$$\sigma_l(\tau,p) = ((0,\tau/p),\Lambda_{-\rho(p)}). \tag{5.26}$$

Left coherent states are then defined by

$$(U_l(\sigma_l(\tau,p))\zeta)(k) = e^{i(\tau/p)k}\zeta(k/p) \qquad \zeta \in \mathcal{H}_l \tag{5.27}$$

and the admissibility condition is now $\zeta \in \mathcal{D}(H_l^{-1/2})$. Collecting together all this information we finally obtain a resolution of the identity in the Krein space (A.20)

$$\begin{aligned} I = |v)(v| + |\chi)(\chi| &+ \frac{1}{c_{\sigma_l}(\phi)} \int_{\Gamma_l} |U_l(\sigma_l(\tau,p))\phi\rangle\langle U_l(\sigma_l(\tau,p))\phi| \, d\mu_l(\tau,p) \\ &+ \frac{1}{c_{\sigma_r}(\psi)} \int_{\Gamma_r} |U_r(\sigma_r(\tau,p))\psi\rangle\langle U_r(\sigma_r(\tau,p))\psi| \, d\mu_r(\tau,p) \end{aligned} \tag{5.28}$$

with $\phi \in \mathcal{H}_l$, $\psi \in \mathcal{H}_r$.

An interesting feature of the sets of coherent states, (5.21) and (5.27), is the fact that they are exactly identical to *wavelets*, i.e. the coherent states of the affine group. Indeed, we may easily convince ourselves that the previous coherent states coincide with the wavelets given for instance in [31] making the following identifications:

$$\frac{1}{p} = a \qquad \frac{\tau}{p} = -b \qquad \frac{\zeta(\omega)}{\sqrt{\omega}} = \hat{\varphi}(\omega). \tag{5.29}$$

Note that the admissibility condition $\zeta \in \mathcal{D}(H_r^{-1/2})$ becomes

$$\int_0^\infty \frac{dk}{k^2}|\zeta(k)|^2 = \int_0^\infty \frac{d\omega}{\omega}|\hat{\varphi}(\omega)|^2 < \infty. \tag{5.30}$$

The invariant measure now reads

$$d\mu(\tau,p) = \frac{d\tau \, dp}{p} = \frac{da \, db}{a^2}. \tag{5.31}$$

Define now the following operator:

$$\mathcal{U} : \mathcal{H}_r = L^2(\mathbb{R}_+) \qquad \frac{dk}{k} \to L^2(\mathbb{R}_+, dk) \tag{5.32}$$

$$(\mathcal{U}\psi)(k) = \hat{\psi}(k) = \psi(k)/\sqrt{k}. \tag{5.33}$$

Then we obtain

$$\frac{1}{\sqrt{c_{\sigma_r}(\zeta)}}\langle U_r(\sigma_r(\tau,p))\zeta, \psi\rangle_{\mathcal{H}_r} = \int_0^\infty e^{-i(\tau/p)k}\bar{\zeta}(k/p)\psi(k)\frac{dk}{k}$$

$$= \sqrt{\frac{a}{C_\varphi}}\int_0^\infty e^{ib\omega}\bar{\hat{\varphi}}(a\omega)\hat{\psi}(\omega)\,d\omega \tag{5.34}$$

which is exactly the wavelet transform for a progressive wavelet φ (cf [31, section 3.1]).

There is another interesting analogy; it is known that the affine group manifold, namely the Poincaré half-plane $a > 0, b \in \mathbb{R}$ is a phase space (this situation is in fact general; for any dimension n, the parameter space of n-dimensional wavelets has the structure of a phase space, see [32] for a discussion of the case n=2). The same phase space is recovered by looking at the coadjoint orbits of the Poincaré group.

The identification between massless coherent states and wavelets opens interesting perspectives. So far (one-dimensional) wavelets have only been considered as with coherent states associated to the '$ax + b$' group, and thus have been used in various problems of classical non-relativistic signal analysis. Now we see that the same wavelets are also particular coherent states of the $(1+1)$-dimensional Poincaré group, corresponding to the massless Wigner representation (see also [21, 7]). This suggests that they could find applications in $(1 + 1)$-dimensional quantum field theory and, more generally, in conformal field theory.

In that context it is also important to understand the relationship between wavelets and coherent states associated with massless representations of the two-dimensional de Sitter groups $SO_0(1,2)$ and the corresponding conformal group $SO_0(2,2)$ [4, 33]). Work in this direction is in progress.

Acknowledgments

UM thanks the Institut de Physique Théorique, Université Catholique de Louvain for financial support. Both authors thank S Twareque Ali, S DeBièvre and J-P Gazeau for stimulating discussions on coherent states. They also thank the referee for some clarifications.

Appendix. Representations of $\mathcal{P}_+^\uparrow(1,1)$

The aim of this appendix is to derive the Wigner representations of $\mathcal{P}_+^\uparrow(1,1)$ corresponding to $m > 0$ by a method which allows the extension to the case $m = 0$. This method is nothing other than an application of Wightman's reconstruction theorem [13]. The key point of this construction consists in finding a Poincaré invariant and positive definite two-point distribution $\mathcal{W}_m(x,y)$ which is solution of the Klein–Gordon equation

$$(\Box + m^2)\mathcal{W}_m = 0. \tag{A.1}$$

Because of Poincaré invariance we get that $\mathcal{W}_m(x,y) = W_m(x-y)$. Furthermore we require $W_m(\xi)$ to be a positive definite distribution and its Fourier transform to have support contained in the future cone. Let $\mathcal{S}(\mathbb{R}^2)$ denote the Schwartz space of infinitely differentiable functions with fast decrease at infinity. There is a natural representation of $\mathcal{P}_+^\uparrow(1,1)$ on $\mathcal{S}(\mathbb{R}^2)$ given by

$$(U(a,\Lambda)f)(x) = f(\Lambda^{-1}(x-a)). \tag{A.2}$$

By Fourier transform we obtain the dual representation

$$(U(a,\Lambda)\tilde{f})(k) = \mathrm{e}^{\mathrm{i}ka}\tilde{f}(\Lambda^{-1}k) \tag{A.3}$$

where we have defined

$$\tilde{f}(k) = \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}^2x \tag{A.4}$$

and kx is the Minkowski inner product $kx = k^0x^0 - k^1x^1$. Now we may introduce an inner product in $\mathcal{S}(\mathbb{R}^2)$ by the following definition

$$\langle f,g\rangle_m = \int \bar{f}(x)\,W_m(x-y)g(y)\,\mathrm{d}^2x\,\mathrm{d}^2y. \tag{A.5}$$

An explicit expression of the two-point function $W_m(x-y)$ is obtained simply by taking the Fourier transform of (A.1)

$$(k^2 - m^2)\widetilde{W}(k) = 0. \tag{A.6}$$

Taking into account the required support properties of $\widetilde{W}$ we obtain

$$\widetilde{W}(k) = c\theta(k^0)\delta(k^2 - m^2) \tag{A.7}$$

and therefore

$$\langle f,g\rangle_m = \int \bar{\tilde{f}}(k)\tilde{g}(k)\,|_{k^0=\sqrt{k^{1^2}+m^2}}\,\frac{\mathrm{d}k^1}{\sqrt{k^{1^2}+m^2}}. \tag{A.8}$$

or

$$\langle f,g\rangle_m = \int_{k^0=\sqrt{k^{1^2}+m^2}} \bar{\tilde{f}}(k)\tilde{g}(k)\frac{\mathrm{d}k^1}{k^0}. \tag{A.9}$$

The Wightman ideal is defined by

$$\mathcal{I}_{\mathrm{W}}^m = \{f \in \mathcal{S}(\mathbb{R}^2) : \langle f,g\rangle_m = 0, \forall g \in \mathcal{S}(\mathbb{R}^2)\}. \tag{A.10}$$

The inner product (A.8) depends only on equivalence classes of $\mathcal{D} = \mathcal{S}(\mathbb{R}^2)/\mathcal{I}_W^m$ and it is not degenerate on $\mathcal{D}$. It is clear that the representation (A.3) carries equivalence classes into equivalence classes and therefore induces a representation of $\mathcal{P}_+^\uparrow(1,1)$ on $\mathcal{D}$. The completion of the set $\mathcal{D}$ in the topology defined by the inner product (A.8) gives the Hilbert space

$$\mathcal{H}_m = L^2(\mathcal{V}_m^+, \mathrm{d}k^1/k^0) \tag{A.11}$$

where $\mathcal{V}_m^+$ is the forward mass hyperbola, defined by

$$\mathcal{V}_m^+ = \left\{(k^0, k^1) \in \mathbb{R}^2 : k^0 = \sqrt{k^{1^2} + m^2}\right\}. \tag{A.12}$$

Now the representation (A.3) may be extended to the whole $\mathcal{H}_m$ and we obtain the following unitary irreducible representation of $\mathcal{P}_+^\uparrow(1,1)$ on $\mathcal{H}_m$:

$$(U_m(a, \Lambda)\psi)(k) = \mathrm{e}^{ika}\psi(\Lambda^{-1}k) \tag{A.13}$$

which is exactly the Wigner representation of mass m.

The extension of this method to construct a representation of $\mathcal{P}_+^\uparrow(1,1)$ in the zero mass case is not immediate. Indeed, as is well known [8], there does not exist any Poincaré invariant positive-definite two-point distribution $\mathcal{W}$ satisfying the equation $\Box\mathcal{W} = 0$ and such that

$$\mathrm{supp}\ \widetilde{W}(k) = C_+ = \{k \in \mathbb{R}^2 : k^\mu k_\mu = 0, k^0 \geqslant 0\}. \tag{A.14}$$

However, if we relax the positivity condition, we may find a Poincaré invariant distribution having the desired support properties, namely

$$W(\xi) = -\frac{1}{4\pi}\log(-\xi^2 + \mathrm{i}\epsilon\xi_0). \tag{A.15}$$

This distribution is not positive-definite and therefore the previous construction gives only a non-degenerate sesquilinear form on the corresponding $\mathcal{D}$

$$\langle f, g\rangle = \int \bar{f}(x)W(x-y)g(y)\,\mathrm{d}^2x\,\mathrm{d}^2y. \tag{A.16}$$

As before $\mathcal{D}$ carries a representation of $\mathcal{P}_+^\uparrow(1,1)$ given by (A.3). In order to obtain a representation on a Hilbert space we must add a new ingredient; we have to complete $\mathcal{D}$ using a Hilbert majorant topology [14, 34], i.e. a Hilbert inner product $(\cdot,\cdot)$ defined on $\mathcal{D}$ such that

$$|\langle f, g\rangle| \leqslant \|f\|\|g\| \qquad \text{where } \|f\|^2 = (f, f). \tag{A.17}$$

The explicit construction runs as follows [12, 15]. Let $\chi \in \mathcal{S}(\mathbb{R}^2)$ such that $\tilde{\chi}(0) = 1$, $\langle\chi, \chi\rangle = 0$. Given $f \in \mathcal{S}(\mathbb{R}^2)$ define

$$f_0(x) = f(x) - \tilde{f}(0)\chi. \tag{A.18}$$

Then the desired inner product may be written as

$$(f,g) = \langle f_0, g_0\rangle + \langle f,\chi\rangle\langle\chi,g\rangle + \bar{\bar{f}}\,(0)\bar{g}(0). \tag{A.19}$$

It may be directly verified that (A.19) defines a Hilbert majorant topology. We may complete $\mathcal{D}$ in this topology and obtain the Krein space [34] (actually it is a Pontriagin space)

$$\mathcal{H} = L^2(C_+, \mathrm{d}k^1/|k^1|) \oplus V \oplus X \tag{A.20}$$

where X is the one-dimensional subspace generated by the function χ while V is the one-dimensional subspace generated by the vector v which is defined (by the Riesz lemma) as the representative of the functional

$$f \mapsto \langle\chi, f\rangle = (v, f). \tag{A.21}$$

There exists a bounded self-adjoint operator η such that

$$\langle f, g\rangle = (f, \eta g) \tag{A.22}$$

and it may be proved that $\eta^2 = 1$ (Krein topology). Now the representation (A.3) extends to a representation U of $\mathcal{P}_+^\uparrow(1,1)$ defined on a dense subset of $\mathcal{H}$. Several remarks are in order. First of all we stress that this representation is not unitary but only η-unitary, i.e. $\langle U(a,\Lambda)f, U(a,\Lambda)g\rangle = \langle f,g\rangle$ but $(U(a,\Lambda)f, U(a,\Lambda)g) \neq (f,g)$. Second, the representation U is neither irreducible nor completely reducible, but non-decomposable. As a final remark we notice that the Hilbert subspace L^2 may be decomposed into the following direct sum

$$L^2(C_+, \mathrm{d}k^1/|k^1|) = \mathcal{H}_\mathrm{l} \oplus \mathcal{H}_\mathrm{r} \tag{A.23}$$

$$\mathcal{H}_\mathrm{l} = L^2(\mathbb{R}_-, \mathrm{d}k^1|k^1|) \qquad \mathcal{H}_\mathrm{r} = L^2(\mathbb{R}_+, \mathrm{d}k^1/k^1) \tag{A.24}$$

(left and right Hilbert spaces) and correspondingly we may quotient the representation U and obtain two unitary irreducible representations $U_{\mathrm{l(r)}}$ defined on $\mathcal{H}_{\mathrm{l(r)}}$. This amounts to considering the matrix elements

$$\langle\psi_1, U(a,\Lambda)\psi_2\rangle_{\mathcal{H}_{\mathrm{l(r)}}} \qquad \psi_1,\psi_2 \in \mathcal{H}_{\mathrm{l(r)}} \tag{A.25}$$

and associating with the sesquilinear forms so defined the operators $U_{\mathrm{l(r)}}(a,\Lambda)$. The final result is

$$(U(a,\Lambda)_{\mathrm{l(r)}}\psi)(k) = \mathrm{e}^{\mathrm{i}ka}\psi(\Lambda^{-1}k) \qquad \psi \in \mathcal{H}_{\mathrm{l(r)}}. \tag{A.26}$$

These are the representations of $\mathcal{P}_+^\uparrow(1,1)$ that are used in the construction of systems of massless coherent states (but the resolution of the identity that we obtain in the end lives in the Krein space H).

References

[1] Klauder J R and Skagerstam B S 1985 *Coherent States—Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[2] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[3] Ali S T and Antoine J-P 1989 *Ann. Inst. H. Poincaré* **51** 23
[4] Ali S T, Antoine J-P and Gazeau J-P 1990 *Ann. Inst. H. Poincaré* **52** 90
[5] Ali S T, Antoine J-P and Gazeau J-P 1991 *Ann. Inst. H. Poincaré* **55** 829
[6] Ali S T, Antoine J-P and Gazeau J-P 1991 *Ann. Inst. H. Poincaré* **55** 857
[7] Kaiser G and Streater R F 1992 Windowed radon transforms, analytic signals, and the wave equation *Wavelets: A Tutorial in Theory and Applications* ed C K Chui (London: Academic) pp 399–441
[8] Wightman A S 1964 Introduction to some aspects of quantized fields *Lecture Notes, Cargèse Summer School* (New York: Gordon and Breach)
[9] Klaiber B 1967 The Thirring model *Quantum Theory and Statistical Physics (Lectures in Theoretical Physics XA)* ed A O Barut and W E Brittin (New York: Gordon and Breach) pp 141–76
[10] Bogoliubov N N, Logunov A A, Oksak A K and Todorov I T 1987 *General Principles of Quantum Field Theory* (Moscow: Nauka) ch 11
[11] Antoine J-P 1979 Indefinite metric and Poincaré covariance in quantum field theory *Mathematical Aspects of Quantum Field Theory (Acta Universitas Wratislaviensis 519)* ed W Karkowski, vol 1, pp 220–65, Wroclaw, 1979
[12] Morchio G, Pierotti D and Strocchi F 1992 *ISAS Preprint 84/87/mp*; 1990 *J. Math. Phys.* **31** 1467
[13] Streater R F and Wightman A S 1964 *PCT, Spin and Statistics and All That* (Reading, MA: Benjamin)
[14] Morchio G and Strocchi F 1980 *Ann. Inst. H. Poincaré* **33** 251
[15] Moschella U 1990 *J. Math. Phys.* **31** 2480
[16] Ali S T and Goldin G A 1991 Quantization, coherent states and diffeomorphism groups *Differential Geometry, Group Representations and Quantization (Lecture Notes in Physics 379)* ed J D Hennig, W Lücke and J Tolar (Berlin: Springer) pp 147–78
[17] Steenrod N 1951 *The Topology of Fibre Bundles* (Princeton: Princeton University Press)
[18] Yaffe L 1982 *Rev. Mod. Phys.* **54** 407
[19] Ali S T 1985 *Riv. Nuovo Cim.* **8** 1
[20] Kirillov A A 1976 *Elements of the Theory of Representations* (Berlin: Springer)
[21] Paul T 1984 *J. Math. Phys.* **25** 3252; 1985 Ondelettes et Mécanique Quantique *Thèse d'Etat* Université d'Aix-Marseille II
[22] Gaal S A 1973 *Linear Analysis and Representation Theory* (Berlin: Springer)
[23] Torrésani B 1991 *J. Math. Phys.* **32** 1273
[24] Torrésani B 1992 *Ann. Inst. H. Poincaré* **56** 215
[25] Guillemin V and Sternberg S 1984 *Symplectic Techniques in Physics* (Cambridge: Cambridge University Press)
[26] Perroud M 1983 *J. Math. Phys.* **24** 1381
[27] Bertrand J and Bertrand P 1990 A class of affine Wigner functions with extended covariance properties *Preprint* LPTM-Paris VII
[28] Arens R 1971 *J. Math. Phys.* **12** 2415
[29] Bacry H 1988 *Localizability and Space in Quantum Physics (Lecture Notes in Physics 304)* (Berlin: Springer)
[30] DeBièvre S 1989 *J. Math. Phys.* **30** 1401
[31] Grossmann A, Morlet J and Paul T 1986 *Ann. Inst. H. Poincaré* **45** 293
[32] Antoine J-P, Carrette P, Murenzi R and Piette B 1993 Image analysis with two-dimensional continuous wavelet transform *Signal Proc.* **31** to appear
[33] DeBièvre S and Gradechi A M El 1993 Quantum Mechanics and coherent states on the anti-de Sitter spacetime and their Poincaré contraction *Ann. Inst. H. Poincaré* to appear
[34] Bognar J 1974 *Indefinite Inner Product Spaces* (Berlin: Springer)